On Φ-Pseudo-Valuation Rings

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1 INTRODUCTION

Throughout this paper, all rings are commutative with identity and if R is a ring, then Z(R) denotes the set of zerodivisors of R and Nil(R) denotes the set of nilpotent elements of R. Our main purpose is to provide another generalization of pseudo-valuation domains (as introduced in [10]) to the context of arbitrary rings (with Z(R) possibly nonzero). Recall from [10] that an integral domain R with quotient field K is called a pseudo-valuation domain (PVD) in case each prime ideal P of R is strongly prime (or a strong prime), in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. Anderson, Dobbs, and the author in [7] generalized the study of pseudo-valuation domains to the context of arbitrary rings. Recall from [7] that a prime ideal P of a ring R is said to be strongly prime (or a strong prime) if aP and bR are comparable for all $a,b \in R$. If R is an integral domain this is equivalent to the original definition of strongly prime as introduced by Hedstrom and Houston in [10] (cf. [1, Proposition 3.1], [2 Proposition 4.2], and [5, Proposition3]). If each prime ideal of R is strongly prime, then R is called a pseudo-valuation ring (PVR).

2 RESULTS

First, recall from [6] and [8] that a prime ideal of R is called divided if it is comparable to every principal ideal of R; equivalently, if it is comparable to every ideal of R. If every prime ideal of R is divided, then R is called a divided ring.

In the following proposition, we show that if a ring R admits a strongly prime ideal, then Nil(R) is a strongly prime ideal and thus Nil(R) is a divided prime. This result justifies our focus in studying pseudo-valuation rings to be restricted to rings R where Nil(R) is a divided prime.

PROPOSITION 0 Let P be a strongly prime ideal of a ring R. Then the prime ideals of R contained in P are strongly prime and are linearly ordered. In particular, Nil(R) is strongly prime and therefore it is a divided prime.
Proof: Let Q be a prime ideal of R contained in P. By applying the same argument as in the proof of [7, Theorem 2], we conclude that Q is strongly prime. By [7, Lemma 1], P is comparable to every prime ideal of R and the prime ideals of R contained in P are linearly ordered. Hence, Nil(R) is prime and therefore it is strongly prime and divided.

Now we state our definition of ϕ-pseudo-valuation rings.

DEFINITION Let R be a ring such that Nil(R) is a divided prime, let S be the set of nonzerodivisors of R, let $T = R_S$ be the total quotient ring of R, and let $K = R_{Nil(R)}$. Define $\phi : T \longrightarrow K$ by $\phi(a/b) = a/b$ for every $a \in R$ and $b \in S$. Then ϕ is a ring homomrphism from T into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = x/1$ for every $x \in R$. Also, observe that $\phi(R)$ is a subring of K with identity. A prime ideal Q of $\phi(R)$ is called K-strongly prime if $xy \in Q$, $x \in K$, $y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$ is called a K-pseudo-valuation ring (K-PVR). A prime ideal P of R is called ϕ-strongly prime, if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime then R is called a ϕ-pseudo-valuation ring (ϕ-PVR). Observe that Q is a prime ideal of $\phi(R)$ if and only if $Q = \phi(P)$ for some prime ideal P of R, and R is a ϕ-PVR if and only if $\phi(R)$ is a K-PVR.

Throughout this section, R denotes a commutative ring with identity such that Nil(R) is a divided prime. Given a ring R, let K $=R_{Nil(R)}$ and T = R_S, where S is the set of nonzerodivisors of R.

Observe that an integral domain R is a PVD if and only if it is a ϕ-PVR. In fact, in Corollary 7, we show that a PVR (in the sense of [7]) is always a ϕ-PVR. Also, observe that a quasilocal zero-dimensional ring is a ϕ-PVR. The following is an example of a zero-dimensional ϕ-PVR that is not a PVR.

EXAMPLE 1 ([7, Remark 15]) Let K be a field, X,Y, and Z be indeterminates, and R = $K[X,Y,Z] / (X^2,Y^2,Z^2) = K[x,y,z]$. Then R is quasilocal zero-dimensional with maximal ideal Nil(R) = $(X,Y,Z) / (X^2,Y^2,Z^2) = (x,y,z)$; hence R is a ϕ-PVR. However, R is not a PVR since $xz \notin yR$ and $y \notin x$Nil(R).

PROPOSITION 2 For a ring R, we have the following :
(1) Ker (ϕ) is contained in Nil(R).
(2) ϕ(R) is an integral domain if and only if for every nonzero $w \in$ Nil(R) there exists a $z \in Z(R)\setminus$Nil(R) such that $zw = 0$ in R.
Proof: (1). Let $x \in$ Ker(ϕ). Then $x = a/b$ for some $a \in R$ and $b \in S$ such that $\phi(a/b) = a/b = 0/1$ in K. Hence, $za = 0$ in R for some $z \in Z(R)\setminus$Nil(R). Thus, $a \in$ Nil(R) since Nil(R) is prime. Hence, $x = a/b = w \in$ Nil(R) since $b \in S$ and Nil(R) is divided. (2). Suppose that ϕ(R) is an integral domain. Since R/Ker(ϕ) $\approx \phi$(R) and Ker(ϕ) $\subset$ Nil(R), we have Ker(ϕ) = Nil(R), and the claim is now clear. Conversely, since for every nonzero $w \in$ Nil(R) there is a $z \in Z(R)\setminus$Nil(R) such that $zw = 0$ in R, we have Ker(ϕ) = Nil(R). Since Nil(R) is prime and R/Nil $\approx \phi$(R), ϕ(R) is an integral domain.

PROPOSITION 3 For a ring R, we have the following:
(1). Nil(T) = Nil(R) and Nil(K) = Nil(ϕ(R))= ϕ(Nil(R)).
(2). Let $x \in$ Nil(K) and write $x=a/b$ for some $a \in R$ and $b \in R\setminus$Nil(R). Then $a \in$ Nil(R) and $x = a/b = w/1$ in K for some $w \in$ Nil(R).

(3). Let $x \in K$ and write $x = a/b$ for some $a \in R$ and $b \in R\backslash Nil(R)$. If $a/b = i/1$ in K for some $i \in R$, then $b|a$ in R; in particular, $a = (i+w)b$ in R for some $w \in Nil(R)$, and therefore a is contained in every prime ideal of R which contains i.

(4). Let $x \in R$ and $y \in R\backslash Nil(R)$. If $x/1 = y/1$ in K, then $x = uy$ in R for some unit u of R; in particular, $(x) = (y)$ in R.

Proof: (1). Note that $Nil(T) = Nil(R)$ since Nil(R) is a divided prime ideal of R. For the second equality, we only need show that $Nil(K) \subset Nil(\phi(R))$. Let $x \in Nil(K)$ and write $x = a/b$ for some $a \in R$ and $b \in R\backslash Nil(R)$. Since Nil(R) is prime, it follows that $a \in Nil(R)$. Since Nil(R) is a divided prime and $a \in Nil(R)$ and $b \in R\backslash Nil(R)$, $x = a/b = w/1$ for some $w \in Nil(R)$. Thus, $x \in Nil(\phi(R))$. (2). Clear by the proof of (1). (3). Since $a/b = i/1$ in K, $z(a-bi) = 0$ in R for some $z \in R\backslash Nil(R)$. Thus, $a-bi = c \in Nil(R)$ since Nil(R) is prime. Since $b \in R\backslash Nil(R)$ and Nil(R) is a divided prime, $c = wb$ for some $w \in Nil(R)$. Hence, $a-bi = c = wb$. Thus, $a = (i+w)b$.
(4). Since $x/1 = y/1$ in K, $z(x-y) = 0$ in R for some $z \in R\backslash Nil(R)$. Thus, $x-y = w \in Nil(R)$. Once again, since $y \in R\backslash Nil(R)$, $w = dy$ for some $d \in Nil(R)$. Hence, $x-y = w = dy$. Thus, $x = (1+d)y$. Since $1+d$ is a unit of R, the claim is clear.

In light of the above proposition, observe that K is quasilocal, zero-dimensional, and a K-PVR with maximal ideal $Nil(\phi(R))$. In general, let A be a divided ring and I be an ideal of A, and let $R = A/I$. Then K is a K-PVR with maximal ideal $Nil(\phi(Rad(I)/I))$, where Rad(I) is the radical ideal of I in A.

The following result is an analogue of [10, Corollary 1.3] and [7, Lemma 1], also see [4, Proposition 1].

PROPOSITION 4 Let P be a ϕ-strongly prime ideal of R. Then P (resp., $\phi(P)$) is a divided prime. In particular, if R is a ϕ-PVR, then R (resp., $\phi(R)$) is a divided ring and hence is quasilocal.

Proof: Deny. Then for some ideal I of R, there is an $i \in I\backslash P$ and a $p \in P\backslash I$. Since $Nil(R) \subset P$, $i \in R\backslash Nil(R)$. Hence,

$(p/i)(i/1) = p/1 \in \phi(P)$. Since $i/1 \notin \phi(P)$ by Proposition 3(4), $p/i \in \phi(P)$. Hence, $i|p$ in R by Proposition 3(3). Thus, $p \in I$ which is a contradiction.

The following result is an analogue of [10, Theorem 1.4], [2, Proposition 4.8], [4, Propositon2], and [7, Theorem 2].

PROPOSITION 5 1. Let P be a ϕ-strongly prime ideal of R and suppose that Q is a prime ideal of R contained in P. Then Q is ϕ-strongly prime. In particular, R is a ϕ-PVR if and only if some maximal ideal of R is ϕ-strongly prime.
2. Let P be a K-strongly prime ideal of $\phi(R)$. If Q is a prime ideal of $\phi(R)$ contained in P, then Q is K-strongly prime. In particular, $\phi(R)$ is a K-PVR if and only if some maximal ideal of $\phi(R)$ is K-strongly prime.
Proof: (1). Suppose that $xy \in \phi(Q)$ for some $x \in K$ and $y \in K$. If $xy \in Nil(\phi(R))$, then either $x \in Nil(\phi(R)) \subset \phi(Q)$ or $y \in Nil(\phi(R) \subset \phi(Q)$ since K is a K-PVR with maximal ideal $Nil(\phi(R))$. Hence, we may assume that $xy \notin Nil(\phi(R))$ and $x \in K\backslash\phi(R)$. Since $xy \in \phi(P)$ and $x \in K\backslash\phi(R)$, we must have $y \in \phi(P)$. Since $x(y^2/xy) = y \in \phi(P)$ and $x \in K\backslash\phi(R)$, we must have $y^2/xy = p/1 \in \phi(P)$ for some $p \in P$. Thus, $y^2 = (xy)(p/1)$ in K. Since $xy \in \phi(Q)$, $y^2 \in \phi(Q)$. Thus, $y \in \phi(Q)$. (2). Since every prime ideal of $\phi(R)$ is of the form $\phi(G)$ for some prime ideal G of R, the claim is clear.

The following lemma is an analogue of [10, Proposition 1.2]. Since the proof is exactly the same as in [10], we leave the proof to the reader.

LEMMA 6 A prime ideal P of R is ϕ-strongly prime if and only if $x^{-1}\phi(P) \subset \phi(P)$ for every $x \in K\backslash\phi(R)$.

COROLLARY 7 (1). A prime ideal P of R is ϕ-strongly prime if and only if for every $a,b \in R\backslash Nil(R)$, either $a|b$ in R or $aP \subset bP$.
(2). A ring R is a ϕ-PVR if and only if for every $a,b \in R\backslash Nil(R)$, either $a|b$ in R or $b|ac$ in R for every nonunit c of R.
(3). If R is a PVR, then R is a ϕ-PVR.

Proof: (1). Suppose that P is ϕ-strongly prime and $a,b \in R\backslash Nil(R)$ such that a/b in R. Then $b/a \in K\backslash\phi(R)$ by Proposition 3(3). Let $p \in P$. Then $(a/b)(p/1) = q/1$ in K for some $q \in P$ by Lemma 6. Thus, $ap = (q+w)b$ in R for some $w \in Nil(R)$ by Proposition 3(3). Hence, $ap \in bP$ in R. Thus, $aP \subset bP$ in R. Conversely, suppose that for every $a,b \in R\backslash Nil(R)$ either $a|b$ or $aP \subset bP$. Let $x \in K\backslash\phi(R)$. Then $x = b/a$ for some $a,b \in R\backslash Nil(R)$ (observe that $b \notin Nil(R)$ since $Nil(R)$ is divided). Hence, a/b in R by Proposition 3(3). Thus, $aP \subset bP$ in R. Hence, $(a/b)\ \phi(P) \subset \phi(P)$. Thus, P is ϕ-strongly prime by Lemma 6. (2). If R is a ϕ-PVR with maximal ideal M, then the claim is clear by (1). Conversely, since for every $a,b \in R$ either $a|b^n$ or $b|a^m$ for some $n,m \geq 1$, the prime ideals of R are linearly ordered by [5, Theorem 1]. Hence R is quasilocal with maximal ideal M. Once again, the claim is clear by (1). (3). This is clear by [7, Theorem 5].

REMARK 8 It was shown in [7, Theorem 5] that a ring R is a PVR if and only for every $a,b \in R$, either $a|b$ or $b|ac$ for every nonunit c of R. Thus, Corollary 7(2) gives a clear difference between a PVR and a ϕ-PVR.

The first part of the following proposition follows easily since the prime ideals of a divided ring R are linearly ordered and $Z(R)$ is a union of prime ideals of R.

PROPOSITION 9 Let R be a divided ring. Then
(1). $Z(R)$ is a prime ideal of R.
(2). If $x \in T\backslash R$, then $x^{-1} \in T$.
Proof: (2). Let $x = a/b \in T\backslash R$ for some $a \in R$ and $b \in S$. Then $a \in S$ since R is divided. Hence, $x^{-1} = b/a \in T$.

Given an ideal I of R, then $I:I = \{x\in T : xI\subset I\}$ and $\phi(I) : \phi(I) = \{x\in K : x\phi(I) \subset \phi(I)\}$

PROPOSITION 10 Let R be a quasilocal ring with maximal ideal M. Then
(1). R is a ϕ-PVR if and only if $M:M$ is a ϕ-PVR with maximal ideal M.
(2). $\phi(R)$ is a K-PVR if and only if $\phi(M) : \phi(M)$ is a K-PVR with maximal ideal $\phi(M)$.

Proof: (1). Suppose that R is a ϕ-PVR. Let $x \in M{:}M\backslash R$. Then $\phi(x) \in K\backslash\phi(R)$ by Proposition 3(3). Since x is a unit of T by Proposition 9(2), $\phi(x^{-1})\phi(M) = \phi(x)^{-1}\phi(M) \subset \phi(M)$ by Lemma 6. Thus, $x^{-1} \in M{:}M$. Thus, x is a unit of M:M. Hence, M is the maximal ideal of M:M. Thus, M:M is a ϕ-PVR since $\phi(M)$ is K-strongly prime. The converse is clear. (2). This follows by a similar argument to that in (1).

Recall that a ring B is called an overring of R (resp., $\phi(R)$) if $R \subset B \subset T$ (resp., $\phi(R) \subset B \subset K$).

PROPOSITION 11 Suppose that R is a ϕ-PVR with maximal ideal M.
(1). If B is an overring of $\phi(R)$ which contains an element of the form 1/s for some nonunit $s \in R\backslash Nil(R)$, then $x^{-1} \in B$ for every $x \in K\backslash\phi(R)$. Furthermore, B is a K-PVR.
(2). If B is an overring of R which contains an element of the form 1/s for some nonunit $s \in S$, then $x^{-1} \in B$ for every $x \in T\backslash R$. Furthermore B is a ϕ-PVR.
Proof: (1). Suppose that B is an overring of $\phi(R)$ which contains an element of the form 1/s for some nonunit $s \in R\backslash Nil(R)$. Let $x \in K\backslash\phi(R)$. Then $x^{-1}(s/1) \in \phi(M) \subset \phi(R)$ by Lemma 6. Hence, $x^{-1} = (x^{-1}s)/s \in B$ since $s^{-1} \in B$. Now, let N be a maximal ideal of B and $xy \in N$ for some $x,y \in K$ with $x \in K\backslash\phi(R)$. Then $y = x^{-1}(xy) \in N$ since $x^{-1} \in B$. Thus, N is K-strongly prime. Hence, B is a K-PVR. (2). Suppose that B is an overring of R which contains an element of the form 1/s for some nonunit $s \in S$. Then $1/s \in \phi(B)$. Hence, $\phi(B)$ is a K-PVR by (1) and therefore B is a ϕ-PVR. Let $x = a/b \in T\backslash R$ for some $a \in R$ and $b \in S$. Then $x^{-1} = b/a \in T$ by Proposition 9(2). Since b/a in R, $a \mid sb$ in R by Corollary 7(2). Hence, $sb = ga$ in R for some $g \in R$. Thus, $x^{-1} = b/a = g/s \in B$ since $s^{-1} \in B$.

COROLLARY 12 Let R be a ϕ-PVR with maximal ideal M. Then
(1). For every prime ideal P of R, P:P is a ϕ-PVR.
(2). For every prime ideal P of $\phi(R)$, P:P is a K-PVR.
(3). For every prime ideal P of $\phi(R)$, $\phi(R)_P$ is a K-PVR.
Proof: (1). If P is maximal, then the claim follows by Proposition 10. Hence, assume that P is nonmaximal. Since P is divided, P:P either contains an element of the form 1/s

for some nonunit $s \in S$, and in this case $P:P$ is a ϕ-PVR by Proposition 11; or $P:P$ does not contain such an element, and in this case it is a ϕ-PVR since it equals R. (2). This follows by a similar argument to that in (1). (3). Once again, if P is maximal, then $\phi(R)_P = \phi(R)$ is a K-PVR. If P is nonmaximal, then $\phi(R)_P$ contains an element of the form $1/s$ for some nonunit $s \in R\setminus Nil(R)$ and therefore it is a K-PVR by Proposition 11.

Recall that a ring B is called a chained ring if the principal ideals of B are linearly ordered.

PROPOSITION 13 Let R be a ϕ-PVR and let B be an overring of R (resp., $\phi(R)$) which contains an element of the form $1/s$ for some nonunit $s \in S$ (resp., $s \in R\setminus Nil(R)$). Then B is a chained ring if and only if for every $a,b \in Nil(R)$ (resp., $Nil(\phi(R))$) either $a|b$ in B or $b|a$ in B.

Proof: We only need prove the converse. Suppose that B is an overring of $\phi(R)$. Let $x,y \in B$ such that neither $x \in Nil(\phi(R))$ nor $y \in Nil(\phi(R))$ and $x\nmid y$ in B. Then $d = x^{-1}y \in K\setminus\phi(R)$. Hence, $d^{-1} = xy^{-1} \in B$ by Proposition 11. Thus, $x = (xy^{-1})y$ in B. Next, suppose that B is an overring of R. Let $x,y \in B$ such that neither $x \in Nil(R)$ nor $y \in Nil(R)$ and $y\nmid x$ in B. Since each $d \in B\setminus R$ is a unit of B by Proposition 11, we may assume that $x,y \in R$. Since $y\nmid x$ in B, $y\nmid x$ in R, and therefore $x|ys$ in R by Corollary 7(2). Hence, $ys = cx$ for some $c \in R$. Hence, $y = (c/s)x$. Thus, $x|y$ in B since $c/s \in B$.

Given a ring R, then R' denotes the integral closure of R in T, and $\phi(R)'$ denotes the integral closure of $\phi(R)$ in K. The following result is an analogue of [7, Lemma 17 and Theorem 19].

PROPOSITION 14 Let R be a ϕ-PVR with maximal ideal M. Then
(1). $R' \subset M:M$ and R' is a ϕ-PVR with maximal ideal M.
(2). $\phi(R)' \subset \phi(M):\phi(M)$ and $\phi(R)'$ is a K-PVR with maximal ideal $\phi(M)$.

Proof: (1). Let $x \in R'\setminus R$. Then $x^{-1}\notin R$. For, if $x^{-1} \in R$, then $x = 1/s$ for some nonunit $s \in S$ which is impossible by [12, Theorem 15]. Since $x^{-1} \notin R$, $\phi(x^{-1}) \notin \phi(R)$ by Proposition 3(3), and hence $\phi(x)\phi(M) \subset \phi(M)$ by Lemma 6. Thus, $xM \subset M$. Hence, $x \in M:M$ and M is a prime ideal of R′ (observe that if $zw \in M$ for some $z,w \in T$, then either $z \in M$ or $w \in M$ since M is ϕ-strongly prime). Since $R \subset R'$ satisfies the INC condition by [12, Theorem 47], M is the maximal ideal of R. Hence, R′ is a ϕ-PVR. (2). Apply a similar argument as in (1).

Our final result is an analogue of [11, Proposition 2.7], [9, Proposition 4.2], and [7, Theorem 21].

PROPOSITION 15 Let R be a ϕ-PVR with maximal ideal M. Then
(1). Every overring of R is a ϕ-PVR if and only if $R' = M:M$.
(2). Every overring of $\phi(R)$ is a K-PVR if and only if $\phi(R)' = \phi(M):\phi(M)$.
Proof: (1). Let C be an overring of R that does not contain an element of the form 1/s for some nonunit $s \in S$. Then observe that $C \subset M:M$, and use a similar argument as in [7, Theorem 21]. (2). Once again, let C be an overring of $\phi(R)$ that does not contain an element of the form 1/s for some nonunit $s \in R\backslash Nil(R)$. Then observe that $C \subset \phi(M):\phi(M)$, and use a similar argument as in the proof of [7, Theorem 21].

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